

MNOP conjecture & equivariant vertex.

Part 1.

- I. Introduction.
- II. Definition of GW / DT invariants.
- III. $\beta=0$ theory of GW / DT.
- IV. MNOP conjecture.

Part 2.

- V. Equivariant vertex of DT theory

I. Introduction

Enumerative geometry of curve counting.

- What is curve?

- stable maps $f: C \rightarrow X$

- 1-dimensional subschemes

- $Z \in X \hookrightarrow I_2 \in \text{Goh}(X)$

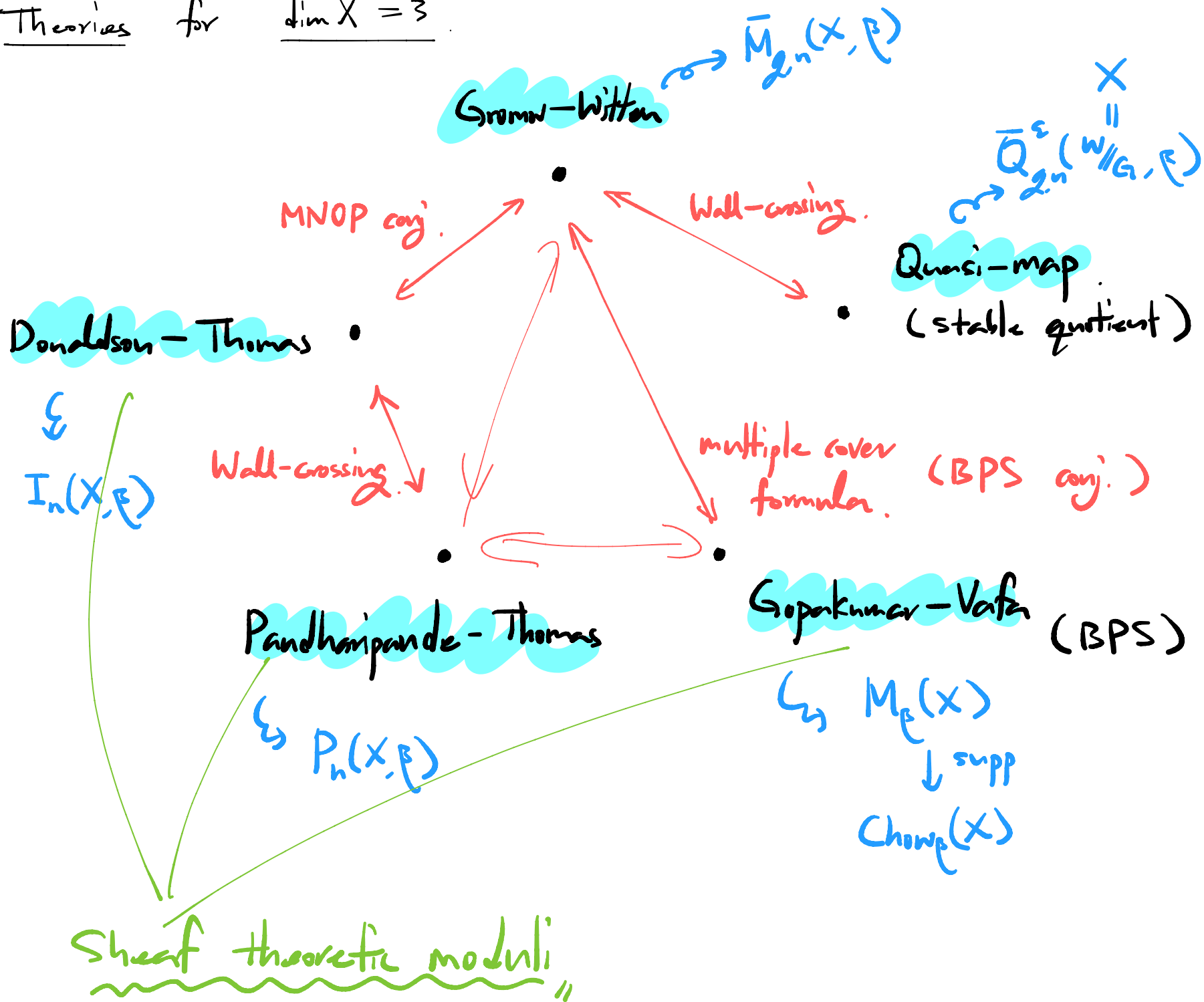
- stable objects in $D^b(X)$

- What is counting? invariants of moduli spaces.

- homological, k-theoretic, motivic, ...

- \mathbb{Q} vs $\mathbb{Z} \xrightarrow{\quad} \text{categorification?}$

* Theories for $\dim X = 3$



II. Definitions of GW/DT invariants.

① GW theory.

X : smooth projective / \mathbb{C} , $\beta \in H_2(X, \mathbb{Z})$

$\overline{M}_{g,r}(X, \beta) \ni [f: (C, p_1, \dots, p_r) \rightarrow X]$: stable r -pointed map
s.t. $g(C) = g$, $f_*[C] = \beta$.

● $T_{\text{an}} = \text{Ext}^0 \left(\{ f^* \Omega_X \rightarrow \Omega_C(\sum p_i) \}, \mathcal{O}_C \right)$

● $\text{Obs} = \text{Ext}^1 \left(\{ f^* \Omega_X \rightarrow \Omega_C(\sum p_i) \}, \mathcal{O}_C \right)$

$\therefore \exists$ natural 2-term pot. of virtual dimension

$$vd = \int_{\beta} c_1(X) + (\dim X - 3) \cdot (1 - g) + r$$

$$\Rightarrow [\overline{M}_{g,r}(X, \beta)]^{\text{vir}} \in A_{vd}(\overline{M}_{g,r}(X, \beta))$$

Using the universal diagram

$$\overline{M}_{g,r}(X, \beta) \xrightarrow{ev_i} X$$

$$\gamma_i \in H^*(X)$$

$$\downarrow \quad \psi_i = c_1(L_i)$$

$$\overline{M}_{g,r}$$

define descendant GW invariants

$$\left\langle \tau_{h_1}(\gamma_1) \cdots \tau_{h_r}(\gamma_r) \right\rangle_{g, \beta}^{GW} := \int [\overline{M}_{g,r}(X, \beta)]^{vir} \prod_{i=1}^r \psi_i^{h_i} \cdot ev_i^*(\gamma_i) \in \mathbb{Q}$$

When $X : \mathbb{C}Y$ 3-fold ($\dim X = 3$, $K_X \simeq \mathcal{O}_X$),

$$vd(\overline{M}_g(X, \beta)) = 0$$

\therefore Define

- $N_{g, \beta} := \int [\overline{M}_g(X, \beta)]^{\text{vir}} 1$
- $Z_X^{\text{GW}}(u)_{\beta} := \sum_{g \geq 0} N_{g, \beta} \cdot u^{2g-2}$

$-e^{\text{top}}(C)$
additive.

- \oplus
- defined for any X .
 - symplectic invariant.
 - mirror symmetry
 - rich structures e.g. CohFT.

- \ominus
- Difficult to compute
 - e.g. $X = \mathbb{P}^1 \Rightarrow$ KdV hierarchy
 - $X = \mathbb{P}^1 \Rightarrow$ Toda hierarchy
 - invariants $\in \mathbb{Q}$
 - multiple cover contributions.
- BPS

② DT theory

X : smooth proj 3-fold.

Interpretation of curve.

GW

vs

DT

- nice curve
- horrible map

• horrible curve

• nice map
(embedding)

$I_n(X, \beta) \ni [Z \subseteq X]$: subscheme of $\dim \leq 1$

s.t. $[Z] = \beta$, $\chi(X, \mathcal{O}_Z) = n$

↗
role of genus.

\exists natural obst. theory of Hilbert scheme $I_n(X, \mathbb{P})$

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$$

$$T_{an} = \text{Ext}^0(I_Z, \mathcal{O}_Z)$$

$$\mathcal{O}_{L_S}^1 = \text{Ext}^1(I_Z, \mathcal{O}_Z)$$

$$\mathcal{O}_{L_S}^2 = \text{Ext}^2(I_Z, \mathcal{O}_Z)$$

$$\cancel{\mathcal{O}_{L_S}^3} = \text{Ext}^3(I_Z, \mathcal{O}_Z) \underset{SD}{=} \text{Hom}(\mathcal{O}_Z, I_Z \otimes k_X)^\vee = 0$$

} 3-term

\rightsquigarrow No $[I_n(X, \mathbb{P})]^{vir} \dots ?!$

*** Key Observation : Consider $I_n(X, \beta)$ as a moduli of stable sheaves.

$$\left\{ \begin{array}{l} \text{subschemes } Z \\ \text{of dim } \leq 1 \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{l} \text{rank 1 torsion free} \\ \text{sheaves } I \text{ s.t. } \det I \simeq \mathcal{O}_X \end{array} \right\}$$

$$\begin{array}{ccc} Z & \longmapsto & I_Z \\ I \hookrightarrow I^\vee \simeq \mathcal{O}_X & \longleftrightarrow & I \end{array}$$

As a moduli of sheaves, \exists natural obst. theory

$$\begin{array}{l} \text{Aut} = \text{Ext}^0(I_Z, I_Z)_0 \\ \bullet \text{ Tan} = \text{Ext}^1(I_Z, I_Z)_0 \\ \bullet \text{ obs}^1 = \text{Ext}^2(I_Z, I_Z)_0 \end{array} \left. \vphantom{\begin{array}{l} \text{Aut} \\ \text{Tan} \\ \text{obs}^1 \end{array}} \right\} \text{ 2-term pot.}$$

$$\text{obs}^2 = \text{Ext}^3(I_Z, I_Z)_0$$

$$\hookrightarrow \because H^0(k_X) \twoheadrightarrow \text{Hom}(I_Z, I_Z k_X)$$

$$\therefore \exists [I_n(X, \rho)]^{\text{vir}} \in A_{\text{vd}}(I_n(X, \rho)) \quad \text{with} \quad \text{vd} = \int_{\rho} c_1(X)$$

↑ independent on n .

Using the universal data

$$\begin{array}{ccc} & I_{\mathbb{Z}} & \\ & \downarrow & \\ I_n(X, \rho) \times X & & \\ \begin{array}{c} p \swarrow \\ I_n(X, \rho) \end{array} & & \searrow q \\ & X & \end{array}$$

$\gamma_i \in H^*(X)$

define the descendent insertions

$$\tau_{k_i}(\gamma_i) := p_* \left(\text{ch}_{k_i+2}(I_{\mathbb{Z}}) \cdot q^* \gamma_i \right).$$

We define descendent DT invariants as

$$\left\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \right\rangle_{\rho, n}^{\text{DT}} := \int [I_n(X, \rho)]^{\text{vir}} \prod_{i=1}^r \tau_{k_i}(\gamma_i) \in \mathbb{Z}.$$

Suppose $X: \mathcal{C}Y$ 3-fold $\Rightarrow v_d = 0$.

\therefore Define $I_{n,p} := \int [I_n(X,p)]^{vir} 1$

$Z_X^{DT}(q)_p := \sum_{n \in \mathbb{Z}} I_{n,p} \cdot q^n$

$\chi(X, \mathcal{O}_Z)$
 \therefore additive.

Moreover, $Tan = Ext^1(I_Z, I_Z)_0 \iff Obs = Ext^2(I_Z, I_Z)_0$
Serre dual.

\therefore Obstruction theory is *symmetric*.

Thm (Behrend) $M: \text{scheme}/\mathbb{C}$ with *symm. obst. theory*.

$$\Rightarrow \int [M]^{vir} 1 = e^{top}(M, v_{Behrend})$$

deformation invariant.

motivic.

⊕ - Integral valued } \Rightarrow motivic refinement.

- Motivic

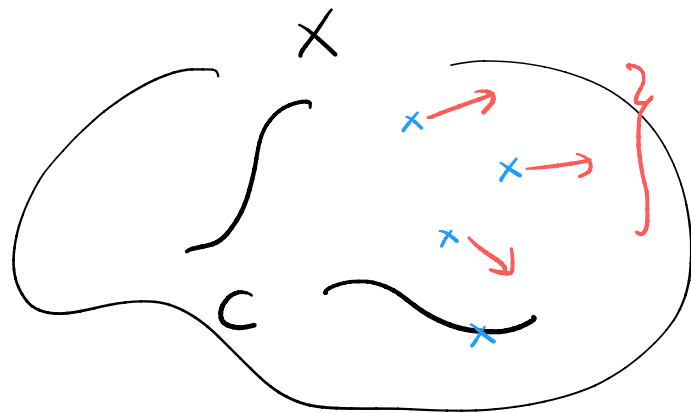
- Higher rank DT theory \rightsquigarrow curve counting?

- Wall-crossing formula.

- Conjectural rationality in q variable.

⊖ - $\dim X = 3$ restriction. (Now we have DT 4-fold)

- Floating points.



} not really curve counting...

\Rightarrow successfully resolved by PT theory.

III. $\beta=0$ theory of GW / DT

① GW theory of $\beta=0$ ($\dim X = 3$)

$$\overline{M}_{g,r}(X, 0) \simeq \overline{M}_{g,r} \times X \xrightarrow{\text{ev}_i = \text{proj}} X$$

constant maps.

$$(3g-3+r \geq 0)$$

$$\begin{aligned} vd &= \int_{\mathbb{P}} c_1(X) + r \\ \text{dim} &= (3g-3+r) + 3 = 3g+r \end{aligned}$$

$\therefore \exists$ obst. bundle of rank $3g \rightsquigarrow$ what?

$T^{\text{vir}} = T - \text{Obs}$

$$\mathcal{O}_{\mathcal{L}_S} = T - T^{\vee}$$

$$= -\text{Ext}^1(\Omega_C(\Sigma_{P_i}), \mathcal{O}_C) + T_{X,x} - \text{Ext}^1(\{f^* \Omega_X \rightarrow \Omega_C(\Sigma_{P_i})\}, \mathcal{O}_C)$$

$$= T_{X,x} - \text{Ext}^1(\mathcal{O}_C \otimes \Omega_{X,x}, \mathcal{O}_C)$$

$f: C \rightarrow X$
constant with
image at x

$$= T_{X,x} - H^1(\mathcal{O}_C) \otimes T_{X,x}$$

$$= T_{X,x} - (\mathbb{C} - H^0(W_C)^\vee) \otimes T_{X,x}$$

$$= H^0(W_C)^\vee \otimes T_{X,x}$$

$$\therefore \mathcal{O}_{\mathcal{L}_S} = E^\vee \otimes T_X \quad \text{where}$$

$$\begin{array}{ccc} & & W_\pi \\ & \swarrow & \\ \mathcal{L}_{g,r} & & \\ \downarrow \pi & & \\ \overline{M}_{g,r} & & \end{array} \quad \begin{array}{l} E := \pi_* W_\pi \\ \text{Hodge bundle} \end{array}$$

\therefore degree $\beta=0$ GW invariants

= Hodge integrals & $H^*(X)$

$$\textcircled{1} \quad g=0, r=3 : \quad \int_{\overline{M}_{0,3} \times X} \prod_{i=1}^3 ev_i^*(\gamma_i) = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$$

$$\textcircled{2} \quad g=1, r=1 :$$

$$\int_{\overline{M}_{1,1} \times X} \underbrace{e(E^\vee \otimes T_x)}_{\parallel c_3(X) - \lambda_1 \cdot c_2(X)} \cup ev^*(\gamma) = \left(- \int_{\overline{M}_{1,1}} \lambda_1 \right) \cdot \int_X c_2(X) \cup \gamma$$

$\underbrace{\hspace{10em}}_{\parallel \frac{1}{24}}$

$$\textcircled{3} \quad g \geq 2, r=0 :$$

$$\int_{\overline{M}_g \times X} e(E^\vee \otimes T_x) = \left(\frac{(-1)^g}{2} \cdot \int_{\overline{M}_g} \lambda^{g-1} \right) \cdot \int_X c_3(X) - c_1(X) c_2(X)$$

$\underbrace{\hspace{10em}}_{\parallel}$

$$\frac{|B_{2g}|}{2g} \cdot \frac{|B_{2g-2}|}{2g-2} \cdot \frac{1}{(2g-2)!}, \text{ where } \frac{x}{e^x - 1} = \sum_{n \geq 0} B_n \cdot \frac{x^n}{n!}$$

$$\underline{\text{Fact}} : Z_x^{\text{GW}}(u)_{\beta=0} := \exp \left(\sum_{g \geq 2} u^{2g-2} \cdot (-1)^g \int_{[\bar{M}_g]} \lambda_{g-1}^3 \right)^{\frac{1}{2}} \int_x c_3 - c_1 c_2 \quad (1)$$

$$\sim M(-q)^{\frac{1}{2}} \int_x c_3 - c_1 c_2 \quad \text{if } q = -e^{iu} \quad (2)$$

where $(3) \quad M(q) := \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^n} : \text{McMahon function}$

$$= \sum_{\pi: 3d\text{-partitions}} q^{|\pi|}$$

② DT theory of $\beta=0$

$I_n(X, \beta=0) = \text{Hilb}(X, n)$: Hilbert scheme of n points.

Study $n=1$ case : $I_1(X, 0) \simeq X$

$\text{rd} = 0$, $\dim = 3$ $\therefore \exists$ Obs of rank 3.

Claim : $\text{Obs} \simeq \Omega_X \otimes k_X^{-1}$ (for CY 3-fold, $\text{Obs} = \text{Tan}^\vee$)

Assuming the claim, we have

$$I_{1, \beta=0} = \int_X e(\Omega_X \otimes k_X^{-1}) = - \int_X c_3 - c_1 c_2$$

Conjecture : $Z_X^{\text{DT}}(q)_{\beta=0} = \sum_{n=0}^{\infty} I_{n, \beta=0} q^n = M(-q)^{\int_X c_3 - c_1 c_2}$

Now proven for any X

(proof of claim) Universal sequence : $0 \rightarrow I_\Delta \rightarrow \mathcal{O}_{x \times x} \rightarrow \mathcal{O}_\Delta \rightarrow 0$

$$\begin{aligned} T_{I_1(X, \mathcal{F}=0)}^{iv} &= H^i(\mathcal{O}_x) - \text{Ext}_{p_1}^i(I_\Delta, I_\Delta) \\ &= \text{Ext}_{p_1}^i(\mathcal{O}, \mathcal{O}_\Delta) + \text{Ext}_{p_1}^i(\mathcal{O}_\Delta, \mathcal{O}) - \text{Ext}_{p_1}^i(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \\ &= \text{Ext}_{p_1}^i(\mathcal{O}, \mathcal{O}_\Delta) - \text{Ext}_{p_1}^i(\mathcal{O}, \mathcal{O}_\Delta \otimes_{p_2^*} k_x^\vee) - \text{Ext}_{p_1}^i(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \end{aligned}$$

$$= \mathcal{O}_x - k_x^\vee - R_{p_1, *}\text{Ext}^i(\Delta_* \mathcal{O}_x, \mathcal{O}_\Delta)$$

Grothendieck duality. \rightarrow $= \mathcal{O}_x - k_x^\vee - R_{p_1, *} \Delta_* \text{Ext}^i(\mathcal{O}_x, \Delta^! \mathcal{O}_\Delta)$

$$= \mathcal{O}_x - k_x^\vee - \Delta^! \Delta_* \mathcal{O}_x$$

$(L_\Delta^* \mathcal{O}_\Delta) \otimes \omega_\Delta [\dim \Delta]$
 $= -(\underline{L_\Delta^* \mathcal{O}_\Delta}) \otimes k_x^\vee$

$$= \mathcal{O}_x - k_x^\vee + (\Lambda_{-1} \Omega_x) \otimes k_x^\vee$$

$\stackrel{why?}{=} \Lambda_{-1} \Omega_x$

$$= \cancel{\mathcal{O}_x} - \cancel{k_x^\vee} + (\cancel{\mathcal{O}_x} - \Omega_x + \Omega_x^2 - \cancel{k_x}) \otimes k_x^\vee$$

$T_x \leftarrow$

$$= \Omega_x^2 \otimes k_x^\vee - \Omega_x \otimes k_x^\vee \rightarrow \underline{\underline{Obs}}$$



IV. MNOP Conjecture (for X : CY 3-fold)

$$\text{GW theory} \xleftrightarrow{q = -e^{i\eta}} \text{DT theory}$$

To formulate the correspondence of $\beta \neq 0$ theory,
we need to define reduced partition function in both theories.

1) GW side: allow disconnected curve C .

$$Z'_x{}^{\text{GW}}(u, v) := \exp \left(\sum_{\substack{\beta \neq 0 \\ g \geq 0}} N_{g, \beta} \cdot u^{2g-2} \cdot v^\beta \right) = 1 + \sum_{\beta \neq 0} \underbrace{Z'_x(u)_\beta}^{\text{compare}} \cdot v^\beta$$

2) DT side: remove floating point contribution.

$$Z'_x{}^{\text{DT}}(q, v) := \frac{\sum_{\beta \neq 0} I_{n, \beta} \cdot q^n \cdot v^\beta}{\sum_{n \geq 0} I_{n, 0} \cdot q^n} = 1 + \sum_{\beta \neq 0} \underbrace{Z'_x(q)_\beta}^{\text{compare}} \cdot v^\beta$$

Fact: Both of reduced theories can be defined by alternative moduli spaces.

easy $\rightarrow \bullet Z'_x{}^{GW}(n)_\mathbb{P} \leftarrow \overline{M}'_g(X, \mathbb{P})$: possibly disconnected C & f is non-constant in every components.

Hard $\rightarrow \bullet Z'_x{}^{DT}(g)_\mathbb{P} \stackrel{\text{Thm}}{=} Z_x{}^{PT}(g)_\mathbb{P} \leftarrow P_n(X, \mathbb{P})$
 hence integral.

cf) $P_n(X, \mathbb{P})$ parametrizes $[Q_X \xrightarrow{s} F]$ s.t.

- F : pure 1-dim'l sheaf of $[F] = \mathbb{P}$, $\chi(F) = n$
- $\text{coker}(s)$: 0-dim'l.

Conjecture: $X: CY$ 3-fold, $\beta \neq 0$.

1) Rationality: $Z'_x{}^{DT}(q)_\beta$ is a rational function symmetric under $q \leftrightarrow q^{-1}$.

2) GW/DT correspondence:

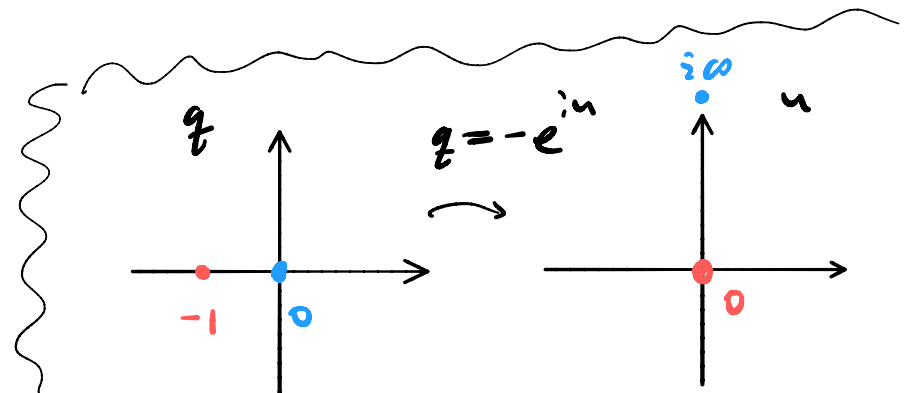
$$Z'_x{}^{GW}(n)_\beta = Z'_x{}^{DT}(q)_\beta \text{ under change of variable } q = -e^{in}.$$

Rmk. 1) proven by Bridgeland building on work of Toda, Joyce.

2) proven for many cases by Pandharipande - Pixton, MOOP.

3) Rationality part is necessary to make sense of $q = -e^{in}$.

$\Rightarrow Z'^{GW}, Z'^{DT}$ is expansion of
a same function at different points.



V. Equivariant vertex of DT theory

① Toric Geometry

$T \curvearrowright X$ toric 3-fold $\rightsquigarrow \Delta$: associated polytope.

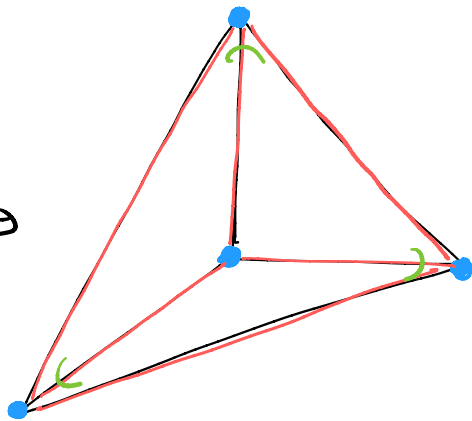
- Vertex of $\Delta \iff \{X_\alpha\}$: T -fixed points.
- Edge of $\Delta \iff \{C_{\alpha\beta}\}$: T -fixed line joining X_α, X_β .

$\dim X = 3 \implies \forall X_\alpha, \exists 3 \text{ edges } C_{\alpha\beta_1}, C_{\alpha\beta_2}, C_{\alpha\beta_3}$.

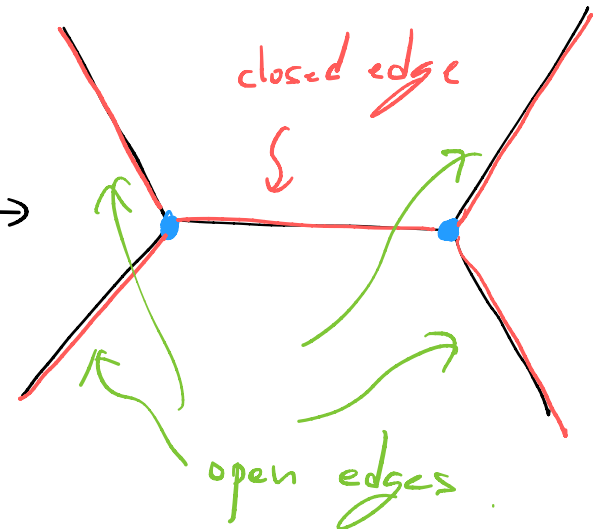
e.g.

\mathbb{P}^3
(A^3)

\iff



$\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \iff$



In fact, above two examples are local pictures of X near $X_\alpha, C_{\alpha\beta}$.

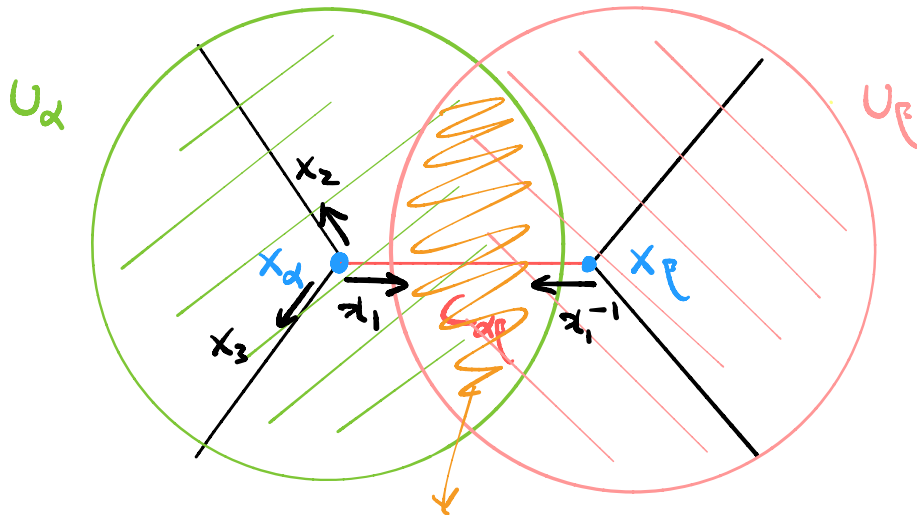
• Near X_α : $\exists U_\alpha \simeq A^3$ centered at X_α .

\exists coordinates t_1, t_2, t_3 for \mathbb{T} , x_1, x_2, x_3 for U_α

s.t. $(t_1, t_2, t_3) \cdot x_i = t_i x_i$.

$$\Rightarrow T_{U_\alpha, X_\alpha} = t_1^{-1} + t_2^{-1} + t_3^{-1}, \quad K_{U_\alpha, X_\alpha} = t_1 t_2 t_3.$$

• Near $C_{\alpha\beta}$: Let $N_{C_{\alpha\beta}/X} \simeq \mathcal{O}_{\mathbb{P}^1}(m_{\alpha\beta}) \oplus \mathcal{O}_{\mathbb{P}^1}(m'_{\alpha\beta})$.



transition map:

$$(x_1, x_2, x_3)$$

\downarrow

$$(x_1^{-1}, x_2 x_1^{-m_{\alpha\beta}}, x_3 x_1^{-m'_{\alpha\beta}})$$

$$U_{\alpha\beta} = U_\alpha \cap U_\beta.$$

② Description of fixed loci $(I_n(X, \rho))^T$

$$Z \in (I_n(X, \rho))^T \Rightarrow Z|_{U_\alpha} : \text{ } T\text{-fixed in } U_\alpha \simeq \mathbb{A}_{x_1, x_2, x_3}^3.$$

$$\Rightarrow I_\alpha := I_Z|_{U_\alpha} \in \mathbb{C}[x_1, x_2, x_3] : \text{monomial ideal}.$$

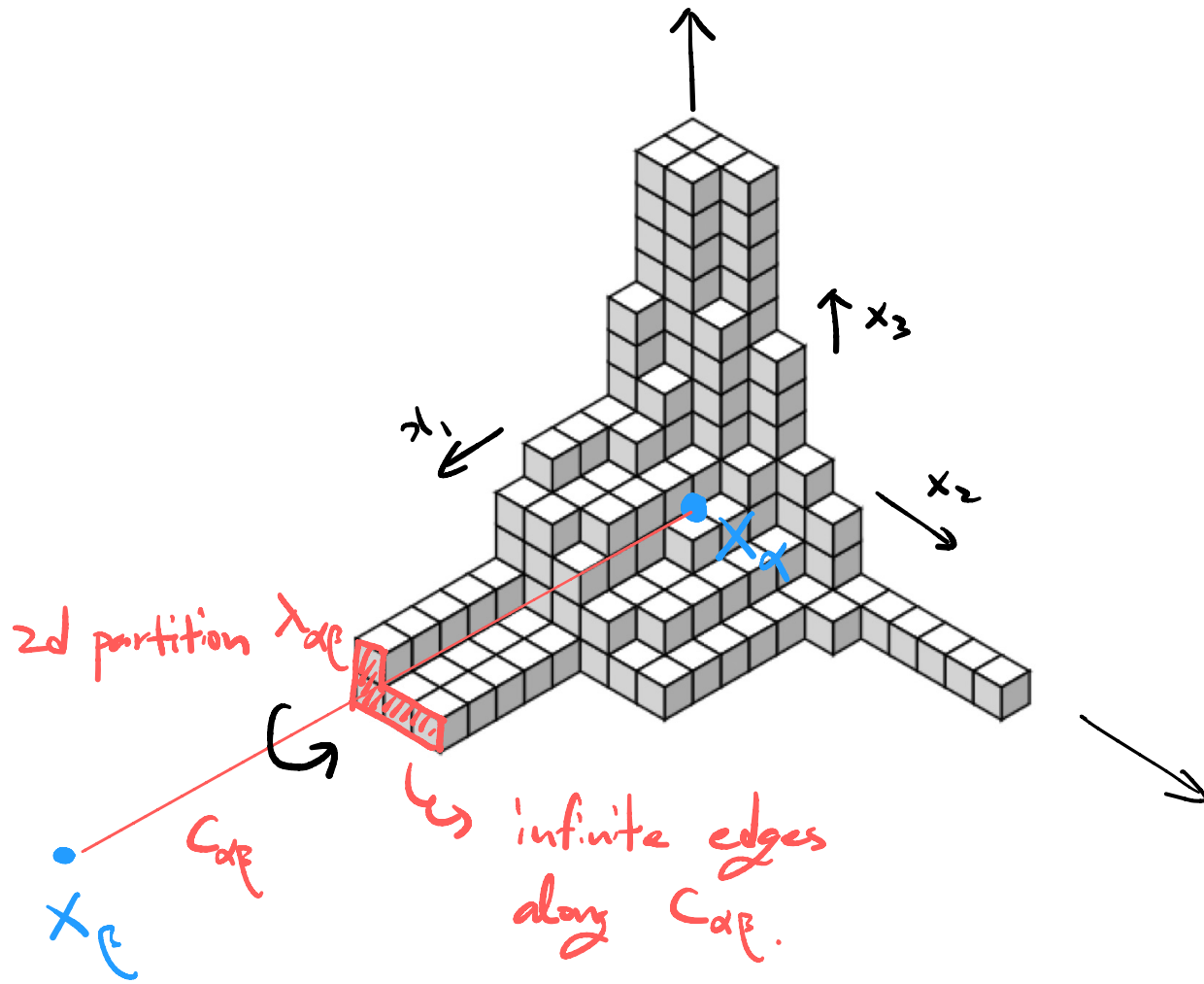
$$\left\{ \begin{array}{l} \text{monomial ideal} \\ I_\alpha \in \mathbb{C}[x_1, x_2, x_3] \end{array} \right\} \longleftrightarrow \left\{ 3\text{d partitions } \pi \right\}$$

$$\pi_\alpha := \left\{ (k_1, k_2, k_3) \mid x_1^{k_1} x_2^{k_2} x_3^{k_3} \notin I_\alpha \right\} \subseteq \mathbb{Z}_{\geq 0}^3$$

Since $Z \subseteq X$ is of $\dim \leq 1$, π_α possibly have infinite legs along 3 edges. Define asymptotic of π_α as

$$\lambda_{\alpha p_1} := \left\{ (k_2, k_3) \mid \forall k_1 \geq 0, x_1^{k_1} x_2^{k_2} x_3^{k_3} \notin I_\alpha \right\} : 2\text{d partition}.$$

e.g.



\therefore Set theoretically,

$$\left(I_n(X, \mathbb{P}) \right)^{\top} \leftrightarrow \begin{cases} \bullet \text{ collection of } \{ \lambda_{\alpha\beta} \} \text{ s.t. } \sum |\lambda_{\alpha\beta}| \cdot [C_{\alpha\beta}] = \mathbb{P} \\ \bullet \text{ collection of } \{ \pi_{\alpha} \} \text{ s.t. whose asymptotics are given by } \lambda_{\alpha\beta}'\text{'s} \end{cases}$$

$$1) f(\alpha, \mathbb{P}) := \sum_{(i,j) \in \lambda_{\alpha\beta}} (m_{\alpha\beta}(i-1) + m_{\alpha\beta}'(j-1) + 1)$$

$$2) |\pi_{\alpha}| := \# \{ \pi_{\alpha} \cap [0, N]^3 \} - (N+1) \sum_{i=1}^3 |\lambda_{\alpha\beta i}| \quad \text{for } N \gg 0.$$

$$\Rightarrow n = \sum |\pi_{\alpha}| + \sum f(\alpha, \mathbb{P})$$

Rmk. $\left(I_n(X, \mathbb{P}) \right)^{\top}$ is also reduced b/c $T_{\text{an}}|_Z = \text{Ext}^1(I_Z, I_Z)_{\text{fix}}^0 \oplus \text{Ext}^1(I_Z, I_Z)_{\text{mov}}^0$

Similarly, $\text{Ext}^2(I_Z, I_Z)_{\text{fix}}^0 = 0$.

③ Virtual localization formula

$$[I_n(X, \rho)]^{\text{vir}} = \sum_{z \in (I_n(X, \rho))^{\top}} \frac{1}{e^{\top}(N_z^{\text{vir}})}$$

Now, $N_z^{\text{vir}} = \chi(\theta) - \chi(I_z, I_z) \in K_{\top}^0(\text{pt})$

We compute N_z^{vir} by local-to-global spectral sequence & Čech cohomology.

* Goal: $N_z^{\text{vir}} = \sum_{\alpha} \tilde{V}_{\alpha} + \sum_{\alpha, \rho} \tilde{E}_{\alpha, \rho}$

↑ vertex contribution
 ↑ Edge contribution.

$$\chi(I_z, I_z) = \sum_{i,j} (-1)^{i+j} H^i(X, \mathcal{E}_{\text{xt}}^j(I_z, I_z))$$

$$= \sum_{i,j} (-1)^{i+j} \check{C}^i(X, \mathcal{E}_{\text{xt}}^j(I_z, I_z))$$

complex \check{C}^{\bullet} computes H^0, H^1, \dots

We use Cech complex w.r.t. $\{U_\alpha\}$ covering.

$$\begin{aligned} \therefore N_Z^{\text{vir}} &= \chi(\mathcal{O}, \mathcal{O}) - \chi(I_Z, I_Z) \\ &= \bigoplus_\alpha \left(\Gamma(U_\alpha) - \sum_i (-1)^i \Gamma(U_\alpha, \text{Ext}^i(I_\alpha, I_\alpha)) \right) \\ &\quad - \bigoplus_{\alpha, \beta} \left(\Gamma(U_{\alpha\beta}) - \sum_i (-1)^i \Gamma(U_{\alpha\beta}, \text{Ext}^i(I_{\alpha\beta}, I_{\alpha\beta})) \right) \end{aligned}$$

\therefore no more $\alpha\beta\gamma \dots$ b/c $I_{\alpha\beta\gamma} \simeq \mathcal{O}_{U_{\alpha\beta\gamma}}$.

$$= \bigoplus_\alpha \left(\underbrace{\chi(\mathcal{O}_{U_\alpha}) - \chi(I_\alpha, I_\alpha)}_{\hookrightarrow \tilde{V}_\alpha} \right) + \bigoplus_{\alpha, \beta} \left(\underbrace{-\chi(\mathcal{O}_{U_{\alpha\beta}}) + \chi(I_{\alpha\beta}, I_{\alpha\beta})}_{\hookrightarrow \tilde{E}_{\alpha\beta}} \right).$$

cf) $\chi(\mathcal{O}_{A^3})$ make sense only as a \mathbb{T} -character.

$$\chi(\mathcal{O}_{A^3}) = \sum_{k_i \geq 0} t_1^{k_1} t_2^{k_2} t_3^{k_3} = \frac{1}{(1-t_1)(1-t_2)(1-t_3)} \quad \text{has pole at } t_i = 1.$$

Useful formula: $\chi(F, G) = \frac{\chi(F)^* \chi(G)}{\chi(\mathcal{O})^*} \quad \forall F, G \in k^{\circ}_{(\mathbb{C}^*)^n}(A^n)$

pf) By $(\mathbb{C}^*)^n$ -equivariant resolution,

$$F = \sum_i \mathcal{O}_{A^n} t^{\vec{d}_i}, \quad G = \sum_j \mathcal{O}_{A^n} t^{\vec{d}_j}$$

$$\chi(F, G) = \sum_{i,j} t^{\vec{d}_j - \vec{d}_i} \cdot \chi(\mathcal{O}_{A^n})$$

$$= \left(\frac{\sum_i t^{\vec{d}_i} \chi(\mathcal{O}_{A^n})}{\chi(\mathcal{O}_{A^n})} \right)^* \cdot \sum_j t^{\vec{d}_j} \chi(\mathcal{O}_{A^n})$$

$$= \frac{\chi(F)^* \chi(G)}{\chi(\mathcal{O})^*}$$

~~W~~

1) Vertex : $0 \rightarrow I_d \rightarrow R_d \rightarrow R_d/I_d \rightarrow 0$

Define $Q_d := \chi(R_d/I_d) = \sum_{(h_1, h_2, h_3) \in \pi_d} t_1^{h_1} t_2^{h_2} t_3^{h_3}$

$\bar{Q}_d(t_1, t_2, t_3) := Q_d(t_1^{-1}, t_2^{-1}, t_3^{-1})$

$$\begin{aligned} \Rightarrow \tilde{V}_d &= \chi(R_d) - \chi(R_d - R_d/I_d, R_d - R_d/I_d) \\ &= \chi(R_d, R_d/I_d) + \chi(R_d/I_d, R_d) - \chi(R_d/I_d, R_d/I_d) \\ &= Q_d + \frac{\bar{Q}_d \cdot \chi(R_d)}{\chi(R_d)^*} - \frac{\bar{Q}_d \cdot Q_d}{\chi(R_d)^*} \end{aligned}$$

$$\begin{aligned} &= Q_d - \frac{\bar{Q}_d}{t_1 t_2 t_3} + Q_d \bar{Q}_d \cdot \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3} \\ \chi(R_d) &= \frac{1}{(1-t_1)(1-t_2)(1-t_3)} \end{aligned}$$

↪ F_d

2) Edge : $0 \rightarrow I_{\alpha\beta} \xrightarrow{\quad} R_{\alpha\beta} \xrightarrow{\quad} R_{\alpha\beta}/I_{\alpha\beta} \rightarrow 0$

$\nwarrow \lambda_{\alpha\beta}$
 $\begin{matrix} \text{SI} \\ \subset [x_1^{\pm}, x_2, x_3] \end{matrix}$

$$\chi(R_{\alpha\beta}/I_{\alpha\beta}) = \underbrace{\sum_{h \in \mathbb{Z}} t_1^h}_{\delta(t_1)} \cdot \underbrace{\sum_{(h_2, h_3) \in \lambda_{\alpha\beta}} t_2^{h_2} t_3^{h_3}}_{:= Q_{\alpha\beta}}$$

Similar calculation gives :

$$\tilde{E}_{\alpha\beta} = \delta(t_1) \cdot \left(-Q_{\alpha\beta} - \frac{\overline{Q}_{\alpha\beta}}{t_2 t_3} + Q_{\alpha\beta} \overline{Q}_{\alpha\beta} \cdot \underbrace{\frac{(1-t_2)(1-t_3)}{t_2 t_3}}_{\hookrightarrow F_{\alpha\beta}} \right)$$

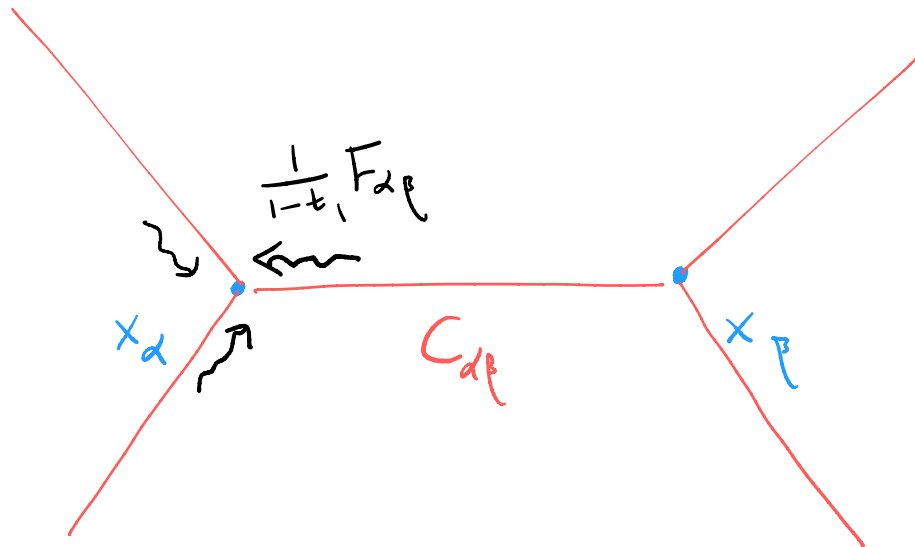
$$\Rightarrow N_z^{\text{vir}} = \sum_{\alpha} F_{\alpha} + \sum_{\alpha, \beta} \delta(t_1) \cdot F_{\alpha\beta}$$

④ Redistribution of vertex & edge contributions.

Goal: redistribute F_α , $\delta(t_1) \cdot F_{\alpha\beta}$ so that each terms are Laurent polynomial ($\mathbb{Q}[t_1^\pm, t_2^\pm, t_3^\pm]$)

Note that
$$\delta(t_1) = (\dots t_1^{-3} + t_1^{-2} + t_1^{-1}) + (1 + t_1 + t_1^2 + \dots)$$

$$= \frac{t_1^{-1}}{1 - t_1^{-1}} + \frac{1}{1 - t_1}$$



$$\therefore \text{ Define } V_\alpha := F_\alpha + \sum_{i=1}^3 \frac{F_{\alpha\beta_i}(t_{i'}, t_{i''})}{1-t_i} \quad \{i, i', i''\} = \{1, 2, 3\}$$

We then need to modify $\tilde{E}_{\alpha\beta}$ by

$$E_{\alpha\beta} := \frac{t_1^{-1} F_{\alpha\beta}(t_2, t_3)}{1-t_1^{-1}} - \frac{F_{\alpha\beta}(t_2 t_1^{-m_{\alpha\beta}}, t_3 t_1^{-m_{\alpha\beta}'})}{1-t_1^{-1}}$$

Clearly, $N_z^{vir} = \sum_\alpha V_\alpha + \sum_{\alpha, \beta} E_{\alpha\beta}$

Thm. $V_\alpha, E_{\alpha\beta}$: Laurent polynomial.

pf) 1) $E_{\alpha, \beta}$: enough to check for t_1 . ($\because \lambda_{\alpha\beta}$: finite partition)

Since numerator vanishes at $t_1=1$, it cancel out the denominator.

2) V_d : enough to observe

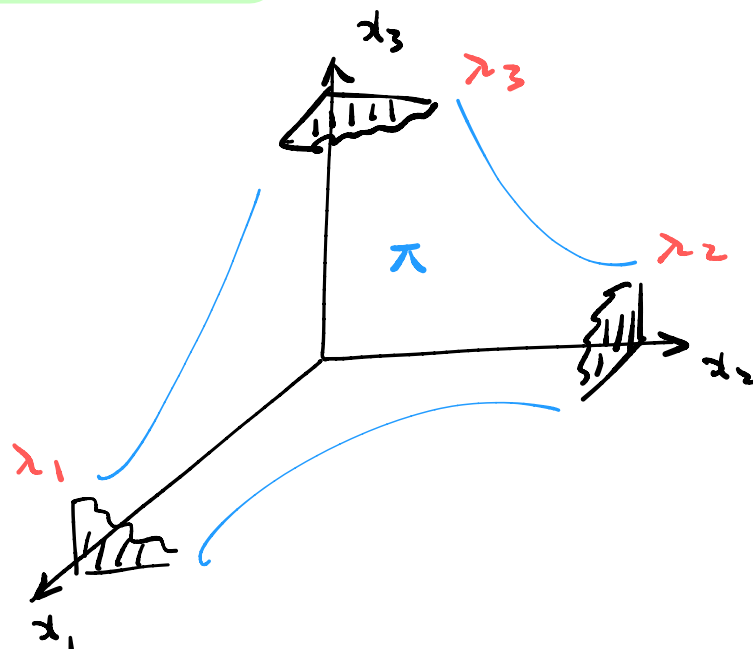
$$Q_d = \frac{Q_{d\beta_1}}{1-t_1} + \frac{Q_{d\beta_2}}{1-t_2} + \frac{Q_{d\beta_3}}{1-t_3} + (\text{Laurent polynomial})$$

///

* Much more can be said if X : toric **CY** 3-fold.

MNOP proves GW/DT correspondence for toric CY 3-fold.

⑤ Equivariant vertex measure.



Define
$$W(\lambda_1, \lambda_2, \lambda_3) := \sum_{\pi \vdash (\lambda_1, \lambda_2, \lambda_3)} \frac{1}{e^{\pi}(V_\pi)} \cdot q^{|\pi|}.$$

Denote
$$s_i := c_i^{\pi}(t_i) \in H^{\pi}(pt).$$

If
$$V_\pi = \sum_{\vec{h} = (h_1, h_2, h_3)} v_{\vec{h}} \cdot t_1^{h_1} t_2^{h_2} t_3^{h_3},$$
 then
$$\frac{1}{e^{\pi}(V_\pi)} = \prod_{(h_1, h_2, h_3)} (h_1 s_1 + h_2 s_2 + h_3 s_3)^{-v_{\vec{h}}}.$$

Thm. (deg 0 DT theory of A^3)

$$W(\underbrace{\phi, \phi, \phi}_{\text{no legs.}}) = M(-q) \cdot \frac{(s_1+s_2)(s_2+s_3)(s_3+s_1)}{s_1 s_2 s_3}$$

local version of
 $\int_X c_3 - c_1 c_2$

where $M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$

∴ $\int_X c_3 - c_1 c_2 = \sum_{\alpha} \frac{c_3 - c_1 c_2}{e(T_{X, X_{\alpha}})} \quad \rightarrow \quad T_{X, X_{\alpha}} = t_1^{-1} + t_2^{-1} + t_3^{-1}$

$$= \sum_{\alpha} \frac{-s_1 s_2 s_3 + (s_1 + s_2 + s_3)(s_1 s_2 + s_2 s_3 + s_3 s_1)}{-s_1 s_2 s_3}$$

$$= \sum_{\alpha} \left(- \frac{(s_1+s_2)(s_2+s_3)(s_3+s_1)}{s_1 s_2 s_3} \right)$$

Opposed to transcendentalty of $W(\phi, \phi, \phi)$, we have

$$\frac{W(1, \phi, \phi)}{W(\phi, \phi, \phi)} = (1 + \frac{q}{L})^{\frac{s_2 + s_3}{s_1}} = \text{rational}$$

reduced \rightarrow "